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Book Reviews

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Featured Review: Introduction to the Foundations of Applied Mathematics. *By Mark H. Holmes.* Springer, New York, 2009. $69.95. xiv+467 pp., hardcover. Texts in Applied Mathematics. Vol. 56. ISBN 978-0-387-87749-5.

This review has two goals. One is to offer a critical review of Mark Holmes's new book on applied mathematics, and the second, broader goal is to put it into the context of the discipline of applied mathematics.

Applied mathematics. Now there are two words that evoke a myriad of definitions, impressions, and prejudices in both mathematicians and nonmathematicians alike! My own favorite was a definition posed by Professor Stefan Drobot at Ohio State when I was a student. In an expository colloquium organized by the graduate students around 1970, he defended, with tongue-in-cheek preciseness, the thesis that "an applied mathematician is one who earns money for doing mathematics." I guess that even makes the great G. H. Hardy an applied mathematician and perhaps removes his need for an apology. Actually, it is often-repeated folklore that Hardy, whose name is part of the Hardy–Weinberg theorem in genetics, often regretted that his name was on such an obvious and trivial result. Over my tenure I have heard mentors and colleagues express notions such as: Once you have studied analysis, algebra, and topology, then you will be ready to study applied mathematics; Pure mathematics is nothing more than parlor games; I wouldn't use paper from the *SIAM Journal of Applied Mathematics* for the bottom of my bird cage! Fortunately, these radical overtones have waned over recent years and most of us view mathematics as the cohesive discipline of "mathematical sciences." This even includes statistics! At the University of Nebraska we have gone from requiring two Ph.D. written comprehensive examinations in algebra and analysis to a flexible system where students may take their exams in any area with the advice and consent of their Ph.D. supervisory committee.

Regarding professions, I have often heard the anecdote that pure mathematics is a young person's game, requiring genius and cleverness, whereas applied mathematicians improve with age as their toolbox fills with tools. For example, it may be hard to imagine a young mathematical scientist determining the strength of an underwater shock wave generated by a nuclear device and its effect on a distant subsurface structure like a submarine; too many tools are required to approach this problem in a serious way. (Perhaps these are just the reminiscences of old applied mathematicians!) And now applied mathematics is becoming everyone's game because even nontraditional "applied" areas, such as abstract algebra, are finding fertile ground in applications.

In a more serious tone, John von Neumann cautioned that there is danger when mathematics loses its grounding in applications, the main one being that it will progress along paths of least resistance. This view might suggest underlying defi-

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nitions of pure and applied mathematics—in pure mathematics the goal is to develop new structures based on previous mathematical structures, whereas applied mathematics seeks to develop new structures based on observations of the natural world. In the latter case, there is an interdependence in which mathematics gives to science and science gives back. The only caveat in these two interpretations is that it is often difficult to discern the differences.

Applied mathematics itself evolved through different paradigms. A paragraph or two, of course, cannot begin to detail these paradigms, but we can get the idea from a few examples. Early in the twentieth century both pure and applied mathematics were spurred by landmark advances in physics. At the same time, both probability theory and synthesis in functional analysis experienced dramatic growth. To many, applied mathematics also meant continuum mechanics, especially during the period around the Second World War when fluid dynamics was a key to modeling flight, detonations and weapon development, and other military and defense strategies. A little later, during the aerospace age, fluids were still king and numerical analysis was born out of the advent of computing devices. But these areas, especially continuum mechanics, spawned an exodus to engineering departments, where aerospace engineering, engineering mechanics, and theoretical and applied mechanics departments blossomed. Mid-century, nonlinear mathematics, again spurred by von Neumann, began to take hold as individuals advanced the understanding of nonlinear equations and processes.

In the 1970s a benchmark paradigm appeared with the publication of Lin and Segel [5], who articulated a radical new viewpoint emphasizing the intimate connections between mathematics and the physical and natural world. They were able to synthesize the thinking of many practitioners on the nature of the subject. Applied mathematics became closely aligned with mathematical modeling and a philosophy that problems in all areas where quantitative ideas occur—economics and finance, physics, biology, and so on—could be formulated as a model. Out of that revolution arose the modeling of practical industrial processes. In fact, in the 1980s everyone wanted to have an "industrial mathematics" program, modeled, for example, on the one at North Carolina State or those at Rensselaer Polytechnic Institute (RPI), or Minnesota. Prior to this, applied mathematics texts and courses often were often organized around traditional mathematical topics such as differential and integral equations, calculus of variations, Green's functions, and boundary value problems, many of which we now consider to belong to the realm of applied functional analysis. Some still hold this traditional view. However, many applied mathematicians now no longer wait for an engineer's knock on the door with a plea to help solve a difficult differential equation; they feel it is their job to learn the engineering and participate in the formulation of the problem as well. These days, beginning perhaps 15–20 years ago with the genome project and advancements in DNA sequencing, mathematical biology is at the forefront, and there are tremendous efforts in life science departments to put more quantitative material in their courses and a rush by mathematics departments to develop joint math-biology programs.

This view, that applied mathematics consists of the development and analysis of "models," has been embraced by every discipline. Lin and Segel extracted the core ideas about how to think about problems and theories, and they articulated what others could not express. Perhaps this is the true essence of the discipline!

As Mark Holmes notes in his new text, changes in the applications of mathematics are the rule. Nudged by advances in technology, funding resources, and so on, the superficial definition of applied mathematics evolves because of these new modeling frontiers. But, expanding upon an observation by Holmes, there is an underlying, unchangeable set of tools that are fundamental to our endeavors. A ticket to success in a particular application is to get in when the gradient is positive, not negative; the best way to do this is to possess the right tools. This makes a book like Holmes's text highly relevant in today's curriculum; even though many of his applications are centered on continuum theories, the book still focuses on core ideas useful for applications. Besides, continuum theories will always be a central part of the applied mathematics paradigm.

Holmes's text was spawned from the Fundamentals of Applied Mathematics (FOAM) course at RPI, which he taught for two years. I was fortunate enough to sit in on that course and hear the lectures of Professor Ash Kapila during a sabbatical at RPI in 1988–1989. This course has a long history that intertwines with the development of applied mathematics beginning in the early 1970s. The cornerstone of FOAM is a two-volume set, the Lin and Segel [5] book noted earlier and a companion volume by Segel and Handelman [8]. The year 1974 was a bumper crop for applied mathematics that year because G. B. Whitham's book [9] on linear and nonlinear waves appeared as well. These classic, landmark books, whose influence on the discipline has been profound, belong on the list of the most important mathematics books published in the twentieth century. Holmes views his new text, in some sense, as an updated version of the Lin and Segel and Segel and Handleman books. That was also the motivation of my own 1987 text on applied mathematics [6], and I suspect there will be additional rewrites of these texts as important areas of application, with yet newer models, emerge. Holmes reasons that applications that are important and current in applied mathematics change over time, requiring revisions.

The view that applied mathematics represents the interplay between mathematics and the natural sciences is underpinned by the notion of a mathematical model. The field of mathematical modeling seeks to create caricatures of physical situations in mathematical or analytical terms, with well-defined quantities (parameters and variables) and relations among these quantities. Important elements of a model are that it is simple, it applies to many situations, and it is predictive. With regard to the latter property, an important part of modeling is to test the model's predictions against real data and fine tune the model accordingly. Unfortunately, nearly all texts written on applied mathematics fail to confront the models they discuss with experimental data. This is certainly true of Holmes's text as well. We all seem to get into the groove of developing theoretical discussions and ask the readers, by faith, to believe in the validity of these models. Validation takes us too far afield and forces us to think about probability, statistics, and data fitting—more topics to include in our already sated textbooks. Typically, and fortunately, however, students do believe our models and are willing to postpone their understanding until these techniques are needed. That this is problematic is well illustrated by the current controversy in physics regarding a "theory" of the cosmos. String theory, as elegant as it may appear, has not been tested by experiment and some believe that it can never be—measuring quantities on the order of the cosmological constant (10^{-119}) seems at present to be outside of the realm of possibility. So, is string theory a valid model? Fluid mechanics, on the other hand, is readily acceptable to us, and to students, because of everyday life experiences. Maybe we really don't need to validate these models in our graduate classes. But one could argue that unfamiliar models of viral infections and the immune system response, some of which we teach in mathematical biology, need justification, and we should not neglect to cite the relevant experimental work.

To ensure that the definition of applied mathematics is not too narrow, I add that it is, in fact, broader than mathematical modeling, and it includes tools and techniques. Clearly, for example, perturbation methods are part of our conception of applied mathematics, but perhaps they are not included in mathematical modeling. Techniques for solving partial differential equations (PDEs), both approximately and exactly, belong to the realm of applied mathematics. Based on the content of Lin and Segel and the FOAM course, it is clear that the RPI conception of applied mathematics does include techniques for solving PDEs. (Kevorkian [4] or Zauderer [10] are good examples that focus on solution techniques.) It may be less clear to many practitioners whether the abstract theory of PDEs counts as applied mathematics in this sense. Certainly, many would argue that PDEs, as presented in books that emphasize only well-posedness, are not "applied mathematics."

What are the important tools of applied mathematics? There are many! A one-year course cannot cover all of them. The key in a course is to present enough basic tools and examples so that students understand how to recognize new problems to which those tools apply and to be able to learn, and possibly discover, new tools on their own. For example, if a student learns about perturbation methods for algebraic and differential equations, then they should be able to pick up a book and understand asymptotic expansions for integrals. The toolbox concept and its metaphors are entertainingly discussed by Mangel [7] in his recent text on mathematical biology.

Holmes has a good selection of important tools. The book begins with two standard chapters on dimensional analysis, scaling, and regular and singular perturbation methods. Many books ignore scaling and its importance in determining the magnitude of various terms in an equation when making approximations; this is a fundamental idea in understanding perturbation methods, and books omitting it should be suspect. Chapter 3 is a very well constructed chapter on kinetics, underpinned by the law of mass action. One question is how to formulate a sequence of elementary reactions that corresponds to a given set of differential equations, for example, the SIR model of disease ecology. Holmes calls this "reverse mass action." In the context of kinetic equations for chemical reactions, he introduces the usual equilibrium definition and stability concepts for systems of ordinary differential equations. He goes on to present singular perturbation and quasi-steady state assumptions in the context of the classic Michaelis–Menten enzyme kinetics model. Another interesting feature is a discussion of how one writes trimolecular and higher order reactions as a sequence of elementary reactions. Recently I used the material in this chapter for my graduate course in mathematical biology.

Diffusion, the subject of Chapter 4, is developed from random walks and Brownian motion. There is some discussion of the continuum description of diffusion and Fourier transforms, and the Langevin equation appears in the last section with a random noise term. There are a few comments about the nature of stochastic differential equations and the fact that noise is a weak derivative of a Brownian motion. Brief appendices discuss stochastic differential equations and Fourier transforms.

Hyperbolic equations and characteristics are introduced in Chapter 5 via an entire chapter on traffic flow. Pedagogically, this is an excellent way to present the key ideas of conservation laws, shock waves, weak solutions, jump conditions, and so on, in a simple context that every student can understand. The extension of these ideas to fluid dynamics appears in later chapters.

Chapters 6 through 9, occupying about 170 pages and over one-third of the book, discuss the dynamics of continua. These four chapters, "Continuum Mechanics in One Spatial Dimension," "Elastic and Viscoelastic Materials," "Continuum Mechanics in Three Dimensions," and "Fluids," show the strong influence of the last part of Lin and Segel [5] and the first part of Segel and Handleman [8]. RPI is a top-notch engineering school, so it may not be surprising that applied mathematics there has evolved along the line of classical continuum mechanics. Ab initio the approach is through Lagrangian, or material, coordinates. A good, long discussion of constitutive laws, involving stress tests, strain, relaxation, and other material properties, accompanies the derivation of the equations of motion. These four chapters would form an excellent, classical course on continuum mechanics for engineers or mathematicians.

There are 20 to 30 problems at the end of each of the nine chapters. These are well chosen; some illustrate material discussed in the text, while many extend the depth and breadth of the topics and applications. A few of the solutions appear on the author's web site.

The author states that the prerequisites for the book are an undergraduate course in differential equations and a calculus sequence that includes some matrix algebra. I would comment, through my own experience in teaching this material for many years, that some multivariable calculus would be helpful, and a level of mathematical maturity, where understanding technical discussions and equation manipulation in the pure and applied sciences is not new, would be beneficial. For example, material on continuum mechanics in higher dimensions requires a fairly sophisticated basis in vector and tensor analysis. Brief appendices cover vector analysis and Taylor's theorem, and an instructor should expect to discuss this material one way or another.

There are seven pages of references with many citations of original articles, where specific applications discussed in the text were published, and of other books, many of which are older, classical texts. Some readers, both instructors and students alike, may pine for more references to collateral material in similar books. As an example, students often have some difficulty understanding singular perturbation methods, and so references where additional worked examples are presented would be helpful. For example, there is no reference to Whitham [9], which influenced generations, or to Keener [3], which has a similar, yet different, approach to applied mathematics. References to Edelstein-Keshet [1], a superbly outstanding and original presentation of mathematical biology, could have opened many eyes to enlightening models in the life sciences, many of the same types that appear in Holmes. Thus, the scope of the references in the text is somewhat limited.

Another tool, perhaps the most important, is the ability to write clearly. We all must have had experience with graduate students who still struggle with, as Mangel [7] puts it, "the culture of bad thinking and bad writing." One way to improve is to read a lot, especially examples of good written communication. In this regard, Holmes's text excels with clear, logical expression. To me, the style leaves just about the right amount of verification to the reader. The book is not clogged with overwhelming detail like some of the 700-page sophomore-level differential equation texts on the market. It seems advisable that, as students progress from one level to the next, their ability to be independent and check salient points on their own must improve. Eventually, they must wade through research articles and research monographs that leave a lot to the reader. The material, in Holmes's words, "becomes more sophisticated as you progress." This provides some flexibility in how the book is used, allowing consideration of breadth and depth. Holmes's text, like his earlier book on perturbations [2], should be accessible to both mathematicians and science and engineering audiences at the senior or beginning graduate level.

In spite of the book's emphasis on continuum theories, it establishes the tools of applied mathematics and the underlying concepts of model development independent of a specific application. The key is for instructors to decide what types of applications they wish to emphasize; if a reasonable component of fluid mechanics is on the list, then Holmes's text would be an outstanding choice, providing a lot of flexibility. In summary, I strongly recommend this book for an introductory senior or graduate level course in applied mathematics.

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Random Graph Dynamics. *By Rick Durrett.* Cambridge University Press, Cambridge, UK, 2007. $64.00. x+212 pp., hardcover. ISBN 978-0-521-86656-9.

Around the year 2000 there was an extraordinary burst of publications associated with phrases like *complex networks, social networks, the web graph, six degrees of separation*, and *small worlds*, which spread rapidly from scientific literature into popular science awareness—Google Scholar shows that survey papers like [1, 6] quickly attracted literally thousands of citations. One aspect of this literature was its analysis of various simple-to-state mathematical models of random graphs, distinct from the classical Erdős–Rényi model much studied in probabilistic combinatorics. The early literature typically involved bare-hands calculations by authors without apparent knowledge of contemporary graduate-level mathematical probability. To people with such knowledge it was immediately clear that well-understood techniques from branching processes, percolation, and interacting particle models were readily applicable to these types of model (see, e.g., your reviewer's own unpolished 2003 lecture notes [2]), so there has been a subsequent steady stream of theorem-proof papers, both by workers switching from the classical Erdős–Rényi school and by those (like the author and reviewer) from mainstream mathematical probability.

As readers of his other books know, Durrett has a particular style—he writes arguments the way one should think directly about them in the first place, rather than via the elegant indirect approaches that often appear in much dug over fields. This is a "lecture notes" style rather than a "reference monograph" style, and indeed the book is an expansion of a Cornell summer workshop course. Following this style, Durrett has chosen a few topics, for each of which which he can develop a few interesting results in a few lectures, making no attempt to cover many topics or to say everything there